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Reduction and transformation formulae for bivariate basic hypergeometric series

C.-Z. Jia *, T.-M. Wang

Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, PR China

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Abstract

The main object of this paper is to establish several bivariate basic hypergeometric series identities by means of elementary series manipulation. Some of them can be applied to yield transformation and reduction formulae for q-Kampé de Fériet functions.

Keywords: Basic hypergeometric series; Kampé de Fériet functions; Jackson's transformation; q-Pfaff–Saalschütz theorem; Sears–Carlitz transformation; Watson's q-Whipple transformation

1. Introduction

For two indeterminates x and q, the shifted factorial is defined by

$$(x;q)_0 = 1 \quad \text{and} \quad (x;q)_n = \prod_{k=0}^{n-1}\bigl(1-q^k x\bigr) \quad \text{with } n = 1, 2, \ldots.$$

When $|q| < 1$, we have the following well-defined infinite product expressions:

$$(x;q)_\infty = \prod_{k=0}^{\infty}\bigl(1-q^k x\bigr) \quad \text{and} \quad (x;q)_n = \frac{(x;q)_\infty}{(q^n x;q)_\infty} \quad \text{for } n \in \mathbb{Z}.$$

* Corresponding author.
E-mail address: cangzhijia@yahoo.com.cn (C.-Z. Jia).

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For the sake of brevity, we also write the factorial product compactly as

$$[a, b, \dots, c; q]_n := (a; q)_n (b; q)_n \cdots (c; q)_n.$$

Following Gasper and Rahman [5], the basic hypergeometric series is defined by

$${}_{1+r}\Phi_s \left[\begin{matrix} a_0, & a_1, & \dots, & a_r \\ & b_1, & \dots, & b_s \end{matrix} \middle| q; z \right] = \sum_{n=0}^{\infty} \left\{ (-1)^n q^{\binom{n}{2}} \right\}^{s-r} \frac{[a_0, a_1, \dots, a_r; q]_n}{[q, b_1, \dots, b_s; q]_n} z^n \tag{1.1}$$

provided that no zero factors appear in the denominator on the right-hand side, i.e., none of the denominator parameters $\{b_k\}_{k=1}^s$ has the form q^{-m} with $m \in \mathbb{N}_0$.

As the q-analogue of Kampé de Fériet function, Srivastava and Karlsson [8, p. 349] defined the generalized bivariate basic hypergeometric function by

$$\Phi^{\lambda:r;s}_{\mu:u;v} \left[\begin{matrix} \alpha_1, \dots, \alpha_\lambda\colon & a_1, \dots, a_r; & c_1, \dots, c_s; & q\colon x, y \\ \beta_1, \dots, \beta_\mu\colon & b_1, \dots, b_u; & d_1, \dots, d_v; & i, j, k \end{matrix} \right] \tag{1.2a}$$

$$= \sum_{m,n=0}^{\infty} \frac{[\alpha_1, \dots, \alpha_\lambda; q]_{m+n}}{[\beta_1, \dots, \beta_\mu; q]_{m+n}} \frac{[a_1, \dots, a_r; q]_m [c_1, \dots, c_s; q]_n}{[b_1, \dots, b_u; q]_m [d_1, \dots, d_v; q]_n} \frac{x^m y^n q^{i\binom{m}{2} + j\binom{n}{2} + kmn}}{(q; q)_m (q; q)_n}. \tag{1.2b}$$

It is not hard to check that when $i, j, k \in \mathbb{N}_0$, the double series $\Phi^{\lambda:r;s}_{\mu:u;v}$ is absolutely convergent for $|x| < 1$, $|y| < 1$ and $|q| < 1$.

In [1], Buschman and Srivastava gave some double hypergeometric series identities with an arbitrary parameter. The corresponding q-analogues were studied by Karlsson [6] and Srivastava and Jain [7]. Recently, Chan et al. [2], Chen and Srivastava [3] and Chu and Srivastava [4] also obtained several double series identities by using quadratic transformations and summation theorems. Motivated by the aforementioned works, this paper will further investigate bivariate q-series and prove several reduction and transformation formulae for q-Kampé de Fériet functions. Some of these q-series identities may be considered as q-analogues of the results that appeared in [2, Eq. (13)], [3, Eqs. (2.1), (2.10), (3.8)] respectively.

The proofs of the theorems all involve, first, a series rearrangement of the form $\sum_{m,n=0}^{\infty} \cdots = \sum_{k=0}^{\infty} \sum_{m}^{k} \cdots$. The opposite step, where we go back to $\sum_{m,n=0}^{\infty} \cdots$, also occurs.

2. Reduction formulae and double series identities

By means of series rearrangements, we now prove two general reduction formulae for double q-series, which will then be applied to derive several double series summation identities.

Theorem 1 *(Reduction formula). For an arbitrary complex sequence $\{\Omega(n)\}_{n=0}^{\infty}$, there holds the following transformation*

$$\sum_{m,n=0}^{\infty} \Omega(m+n) \frac{[a, qa^{\frac{1}{2}}, b; q]_m}{[q, a^{\frac{1}{2}}, qa/b; q]_m} \frac{(1/b^2; q)_n}{(q; q)_n} b^{-2m} x^{m+n} \tag{2.1a}$$

$$= \sum_{n=0}^{\infty} \Omega(n) \frac{(a/b^2; q)_n (1/b; q)_n (-qa^{\frac{1}{2}}/b; q)_n}{(q; q)_n (qa/b; q)_n (-a^{\frac{1}{2}}/b; q)_n} x^n \tag{2.1b}$$

provided that both series displayed above are absolutely convergent.

Proof. Recalling the nearly-poised summation formula [5, Ex. 2.14]

$$ {}_4\Phi_3\left[\begin{matrix} a, & qa^{\frac12}, & b, & q^{-n} \\ & a^{\frac12}, & qa/b, & q^{1-n}b^2 \end{matrix}\middle| q; q\right] = \frac{[a/b^2, 1/b, -qa^{\frac12}/b; q]_n}{[1/b^2, qa/b, -a^{\frac12}/b; q]_n} $$

the expression (2.1a) becomes

$$ \begin{aligned} &\sum_{n=0}^{\infty}\Omega(n)x^n\sum_{m=0}^{n}\frac{(a;q)_m(qa^{\frac12};q)_m(b;q)_m}{(q;q)_m(a^{\frac12};q)_m(qa/b;q)_m}\frac{(1/b^2;q)_{n-m}}{(q;q)_{n-m}}b^{-2m} \\ &\quad=\sum_{n=0}^{\infty}\Omega(n)\frac{(1/b^2;q)_n}{(q;q)_n}x^n\,{}_4\Phi_3\left[\begin{matrix} a, & qa^{\frac12}, & b, & q^{-n} \\ & a^{\frac12}, & qa/b, & q^{1-n}b^2 \end{matrix}\middle| q; q\right] \\ &\quad=\sum_{n=0}^{\infty}\Omega(n)\frac{(a/b^2;q)_n(1/b;q)_n(-qa^{\frac12}/b;q)_n}{(q;q)_n(qa/b;q)_n(-a^{\frac12}/b;q)_n}x^n \end{aligned} $$

which is just (2.1b). This completes the proof of the theorem. □

Specializing $\Omega(n)=1$ in Theorem 1, we get the following transformation between two well-poised ${}_3\Phi_2$-series.

Corollary 2 *(Transformation formula).*

$$ {}_3\Phi_2\left[\begin{matrix} a/b^2, & -qa^{\frac12}/b, & 1/b \\ & -a^{\frac12}/b, & qa/b \end{matrix}\middle| q; x\right] = \frac{(x/b^2;q)_\infty}{(x;q)_\infty}\,{}_3\Phi_2\left[\begin{matrix} a, & qa^{\frac12}, & b \\ & a^{\frac12}, & qa/b \end{matrix}\middle| q; \frac{x}{b^2}\right]. $$

Letting in Theorem 1

$$ x=\frac{qa^{\frac12}}{c}\quad\text{and}\quad\Omega(n)=\frac{(c;q)_n}{(qa/b^2c;q)_n} $$

and then evaluating the corresponding (2.1b) by using [5, II-13]

$$ {}_4\Phi_3\left[\begin{matrix} a, -qa^{\frac12}, b, c \\ -a^{\frac12}, qa/b, qa/c \end{matrix}\middle| q; \frac{qa^{\frac12}}{bc}\right] = \frac{[qa, qa/bc, qa^{\frac12}/b, qa^{\frac12}/c; q]_\infty}{[qa/b, qa/c, qa^{\frac12}, qa^{\frac12}/bc; q]_\infty} $$

we obtain the following summation formula.

Corollary 3 *(Summation formula).*

$$ \begin{aligned} &\Phi^{1:3;1}_{1:2;0}\left[\begin{matrix} c: & a, qa^{\frac12}, b; & 1/b^2; & q: & qa^{\frac12}/b^2c, & qa^{\frac12}/c \\ qa/b^2c: & a^{\frac12}, qa/b; & -; & 0, & 0, & 0 \end{matrix}\right] \\ &\quad=\frac{[qa^{\frac12}, qa^{\frac12}/bc, qa/b^2, qa/bc; q]_\infty}{[qa^{\frac12}/b, qa^{\frac12}/c, qa/b, qa/b^2c; q]_\infty}. \end{aligned} $$

Its terminating case $c=q^{-N}$ reads as follows:

$$ \Phi^{1:3;1}_{1:2;0}\left[\begin{matrix} q^{-N}: & a, qa^{\frac12}, b; & 1/b^2; & q: & q^{N+1}a^{\frac12}/b^2, & q^{N+1}a^{\frac12} \\ q^{N+1}a/b^2: & a^{\frac12}, qa/b; & -; & 0, & 0, & 0 \end{matrix}\right] = \frac{[qa^{\frac12}, qa/b^2; q]_N}{[qa^{\frac12}/b, qa/b; q]_N}. $$

Similarly, letting in Theorem 1

$$x=\frac{qa}{bcd}\quad\text{and}\quad \Omega(n)=\frac{[qa^{\frac{1}{2}}/b,c,d;q]_n}{[a^{\frac{1}{2}}/b,qa/b^2c,qa/b^2d;q]_n}$$

and then reformulating the corresponding (2.1b) by means of the non-terminating very well-poised ${}_6\Phi_5$-summation formula [5, II-20]

$${}_6\Phi_5\left[\begin{matrix}a,qa^{\frac{1}{2}},-qa^{\frac{1}{2}},b,c,d\\ a^{\frac{1}{2}},-a^{\frac{1}{2}},qa/b,qa/c,qa/d\end{matrix}\middle|q;\frac{qa}{bcd}\right]=\frac{[qa,qa/bc,qa/bd,qa/cd;q]_\infty}{[qa/b,qa/c,qa/d,qa/bcd;q]_\infty}$$

we find the following summation formula.

Corollary 4 *(Summation formula).*

$$\Phi^{3:3;1}_{3:2;0}\left[\begin{matrix}qa^{\frac{1}{2}}/b, & c, & d: & a,qa^{\frac{1}{2}},b; & 1/b^2; & q: & qa/b^3cd,qa/bcd\\ a^{\frac{1}{2}}/b, & qa/b^2c, & qa/b^2d: & a^{\frac{1}{2}},qa/b; & -; & & 0,\ 0,\ 0\end{matrix}\right]$$
$$=\frac{[qa/b^2,qa/bc,qa/bd,qa/b^2cd;q]_\infty}{[qa/b,qa/b^2c,qa/b^2d,qa/bcd;q]_\infty}.$$

For the special case $d=q^{-N}$, the above corollary becomes the following terminating summation formula:

$$\Phi^{3:3;1}_{3:2;0}\left[\begin{matrix}qa^{\frac{1}{2}}/b,c,q^{-N}:\ a,qa^{\frac{1}{2}},b;1/b^2;q:\ q^{N+1}a/b^3c,q^{N+1}a/bc\\ a^{\frac{1}{2}}/b,qa/b^2c,q^{N+1}a/b^2:\ a^{\frac{1}{2}},qa/b;-;0,0,0\end{matrix}\right]$$
$$=\frac{[qa/b^2,qa/bc;q]_N}{[qa/b,qa/b^2c;q]_N}.$$

In addition, taking in Theorem 1

$$x=\frac{q^{2+N}a^2}{b^3cde}\quad\text{and}\quad \Omega(n)=\frac{[q^{-N},qa^{\frac{1}{2}}/b,c,d,e;q]_n}{[q^{N+1}a/b^2,a^{\frac{1}{2}}/b,qa/b^2c,qa/b^2d,qa/b^2e;q]_n}$$

and then transforming the corresponding (2.1b) by Watson's q-Whipple transformation formula [5, III-18]

$${}_8\Phi_7\left[\begin{matrix}a, & qa^{\frac{1}{2}}, & -qa^{\frac{1}{2}}, & b, & c, & d, & e, & q^{-n}\\ & a^{\frac{1}{2}}, & -a^{\frac{1}{2}}, & qa/b, & qa/c, & qa/d, & qa/e, & aq^{n+1}\end{matrix}\middle|q;\frac{q^{2+n}a^2}{bcde}\right]$$
$$=\frac{[qa,qa/de;q]_n}{[qa/d,qa/e;q]_n}\,{}_4\Phi_3\left[\begin{matrix}qa/bc, & d, & e, & q^{-n}\\ & qa/b, & qa/c, & q^{-n}de/a\end{matrix}\middle|q;q\right]$$

we derive the following reduction formula.

Corollary 5 *(Reduction formula).*

$$\Phi^{5:3;1}_{5:2;0}\left[\begin{matrix}qa^{\frac{1}{2}}/b,c,d,e,q^{-N}:\ a,qa^{\frac{1}{2}},b;1/b^2;q:\ q^{2+N}a^2/b^5cde,q^{2+N}a^2/b^3cde\\ a^{\frac{1}{2}}/b,qa/b^2c,qa/b^2d,qa/b^2e,q^{N+1}a/b^2:\ a^{\frac{1}{2}},qa/b;-;0,0,0\end{matrix}\right]$$
$$=\frac{[qa/b^2,qa/b^2de;q]_N}{[qa/b^2d,qa/b^2e;q]_N}\,{}_4\Phi_3\left[\begin{matrix}qa/bc, & d, & e, & q^{-N}\\ & qa/b, & qa/b^2c, & q^{-N}b^2de/a\end{matrix}\middle|q;q\right].$$

Our second general reduction formula reads as follows.

Theorem 6 *(Reduction formula). For an arbitrary complex sequence $\{\Omega(n)\}_{n=0}^{\infty}$, there holds the following transformation*

$$\sum_{m,n=0}^{\infty}\Omega(m+n)\frac{(a;q)_m(qa^{\frac{1}{2}};q)_m(-b/qa^{\frac{1}{2}};q)_m}{(q;q)_m(a^{\frac{1}{2}};q)_m(b;q)_m}\frac{(-a^{-\frac{1}{2}};q)_n}{(q;q)_n}\big(-a^{-\frac{1}{2}}\big)^m x^{m+n} \tag{2.2a}$$

$$=\sum_{n=0}^{\infty}\Omega(n)\frac{(-a^{\frac{1}{2}};q)_n(b/qa;q)_n(q^2a/b;q)_n}{(q;q)_n(b;q)_n(qa/b;q)_n}x^n \tag{2.2b}$$

provided that both series displayed above are absolutely convergent.

Proof. According to the transformation [5, Eq. (3.10.9)]

$${}_4\Phi_3\left[\begin{matrix} a,\ qa^{\frac{1}{2}},\ -b/qa^{\frac{1}{2}}, & q^{-n} \\ a^{\frac{1}{2}}, \quad b, & -a^{\frac{1}{2}}q^{1-n}\end{matrix}\,\Big|\,q;q\right]=\frac{[b/qa,-a^{\frac{1}{2}},q^2a/b;q]_n}{[b,-a^{-\frac{1}{2}},qa/b;q]_n}$$

we may reformulate (2.2a) as follows

$$\sum_{n=0}^{\infty}\Omega(n)x^n\sum_{m=0}^{n}\frac{(a;q)_m(qa^{\frac{1}{2}};q)_m(-b/qa^{\frac{1}{2}};q)_m}{(q;q)_m(a^{\frac{1}{2}};q)_m(b;q)_m}\frac{(-a^{-\frac{1}{2}};q)_{n-m}}{(q;q)_{n-m}}\big(-a^{-\frac{1}{2}}\big)^m$$

$$=\sum_{n=0}^{\infty}\Omega(n)\frac{(-a^{-\frac{1}{2}};q)_n}{(q;q)_n}x^n\,{}_4\Phi_3\left[\begin{matrix} a,\ qa^{\frac{1}{2}},\ -b/qa^{\frac{1}{2}}, & q^{-n} \\ a^{\frac{1}{2}}, \quad b, & -a^{\frac{1}{2}}q^{1-n}\end{matrix}\,\Big|\,q;q\right]$$

$$=\sum_{n=0}^{\infty}\Omega(n)\frac{(-a^{\frac{1}{2}};q)_n(b/qa;q)_n(q^2a/b;q)_n}{(q;q)_n(b;q)_n(qa/b;q)_n}x^n$$

which is precisely (2.2b). This completes the proof of this theorem. □

For $\Omega(n)=1$, the last theorem would readily yield the following transformation.

Corollary 7 *(Transformation formula).*

$${}_3\Phi_2\left[\begin{matrix} -a^{\frac{1}{2}}, b/qa, q^2a/b \\ b, \quad qa/b\end{matrix}\,\Big|\,q;x\right]=\frac{(-a^{-\frac{1}{2}}x;q)_\infty}{(x;q)_\infty}\,{}_3\Phi_2\left[\begin{matrix} a, qa^{\frac{1}{2}}, -b/qa^{\frac{1}{2}} \\ a^{\frac{1}{2}}, \quad b\end{matrix}\,\Big|\,q;-\frac{x}{a^{\frac{1}{2}}}\right].$$

3. Further transformation formulae

By manipulating the double q-series, we shall establish in this section six further transformation and reduction formulae.

Theorem 8 *(q-Analogue of transformation [2, Eq. (13)]). For an arbitrary complex sequence $\{\Omega(n)\}_{n=0}^{\infty}$, there holds the following transformation*

$$\sum_{m,n=0}^{\infty}\Omega(m+n)q^{\binom{m+1}{2}}\frac{(\alpha;q)_{m+2n}}{(\gamma;q)_{m+2n}}\frac{x^m}{(q;q)_m}\frac{y^n}{(q;q)_n} \tag{3.1a}$$

$$= \sum_{m,n=0}^{\infty} \Omega(m+n) q^{\binom{m+1}{2}} \frac{(\alpha;q)_{m+2n}(\gamma/\alpha;q)_m(-q^{m+1}\gamma x/\alpha y;q)_n}{(\gamma;q)_{2m+2n}(q;q)_m(q;q)_n} x^m y^n \tag{3.1b}$$

provided that both series displayed above are absolutely convergent.

Proof. Applying Jackson's transformation [5, Ex. 1.16(III)]

$$ {}_2\Phi_1\left[\begin{matrix} q^{-n}, & b \\ & c \end{matrix} \middle| q; z\right] = \frac{(c/b;q)_n}{(c;q)_n} {}_3\Phi_2\left[\begin{matrix} q^{-n}, & b, & q^{-n}bz/c \\ & q^{1-n}b/c, & 0 \end{matrix} \middle| q; q\right] \tag{3.2}$$

we can reformulate (3.1a) as follows:

$$\sum_{k=0}^{\infty} \Omega(k) q^{\binom{k+1}{2}} \frac{(\alpha;q)_k x^k}{(q;q)_k(\gamma;q)_k} {}_2\Phi_1\left[\begin{matrix} q^{-k}, & q^k\alpha \\ & q^k\gamma \end{matrix} \middle| q; -y/x\right]$$

$$= \sum_{k=0}^{\infty} \Omega(k) q^{\binom{k+1}{2}} \frac{(\alpha;q)_k(\gamma/\alpha;q)_k x^k}{(q;q)_k(\gamma;q)_{2k}} {}_3\Phi_2\left[\begin{matrix} q^{-k}, q^k\alpha, -q^{-k}\alpha y/\gamma x \\ q^{1-k}\alpha/\gamma, 0 \end{matrix} \middle| q; q\right]$$

$$= \sum_{k=0}^{\infty}\sum_{n=0}^{k} q^{\binom{k+1}{2}+\binom{n}{2}-kn} \Omega(k) \frac{(\alpha;q)_{k+n}(\gamma/\alpha;q)_{k-n}(-q^{1+k-n}\gamma x/\alpha y;q)_n}{(\gamma;q)_{2k}(q;q)_{k-n}(q;q)_n} x^{k-n} y^n.$$

Finally, going back to a series

$$\sum_{m,n=0}^{\infty} \Omega(m+n) q^{\binom{m+1}{2}} \frac{(\alpha;q)_{m+2n}(\gamma/\alpha;q)_m(-q^{m+1}\gamma x/\alpha y;q)_n}{(\gamma;q)_{2m+2n}(q;q)_m(q;q)_n} x^m y^n$$

by setting $k = m+n$, we just get the formula stated in Theorem 8. □

In the proof of the last theorem, if we apply instead of (3.2), Jackson's transformation [5, Ex. 1.16(II)]

$$ {}_2\Phi_1\left[\begin{matrix} q^{-n}, & b \\ & c \end{matrix} \middle| q; z\right] = \frac{(c/b;q)_n}{(c;q)_n} b^n {}_3\Phi_1\left[\begin{matrix} q^{-n}, & b, & q/z \\ & q^{1-n}b/c & \end{matrix} \middle| q; z/c\right]$$

we obtain another q-analogue of [2, Eq. (13)].

Theorem 9 *(Another q-analogue of transformation [2, Eq. (13)]). For an arbitrary complex sequence $\{\Omega(n)\}_{n=0}^{\infty}$, there holds the following transformation*

$$\sum_{m,n=0}^{\infty} \Omega(m+n) q^{\binom{m+1}{2}} \frac{(\alpha;q)_{m+2n}}{(\gamma;q)_{m+2n}} \frac{x^m}{(q;q)_m} \frac{y^n}{(q;q)_n} \tag{3.3a}$$

$$= \sum_{m,n=0}^{\infty} \Omega(m+n) \frac{(\alpha;q)_{m+2n}(\gamma/\alpha;q)_m(-qx/y;q)_n}{(\gamma;q)_{2m+2n}(q;q)_m(q;q)_n} (\alpha x)^m y^n q^{\frac{m}{2}(1+3m+4n)} \tag{3.3b}$$

provided that both double series displayed above are absolutely convergent.

Similarly, applying in place of (3.2), Jackson's transformation [5, Ex. 1.16(I)]

$$ {}_2\Phi_1\left[\begin{matrix} q^{-n}, & b \\ & c \end{matrix} \middle| q; z\right] = \frac{(q^{-n}bz/c;q)_\infty}{(bz/c;q)_\infty} {}_3\Phi_2\left[\begin{matrix} q^{-n}, & c/b, & 0 \\ & c, & qc/bz \end{matrix} \middle| q; q\right]$$

we establish the following theorem.

Theorem 10 *(Transformation formula). For an arbitrary complex sequence $\{\Omega(n)\}_{n=0}^{\infty}$, there holds the following transformation*

$$\sum_{m,n=0}^{\infty} \Omega(m+n) q^{m(m+n)+\binom{n}{2}} \frac{(\alpha;q)_{m+2n}}{(\gamma;q)_{m+2n}} \frac{x^m}{(q;q)_m} \frac{y^n}{(q;q)_n} \tag{3.4a}$$

$$= \sum_{m,n=0}^{\infty} \Omega(m+n)(-1)^n q^{\binom{m}{2}} \left(\frac{\alpha y}{\gamma}\right)^{m+n} \frac{(\alpha;q)_{m+n}(-q^{1+n}\gamma x/\alpha y;q)_m(\gamma/\alpha;q)_n}{(\gamma;q)_{m+2n}(q;q)_m(q;q)_n} \tag{3.4b}$$

provided that both series displayed above are absolutely convergent.

By appealing to the Sears–Carlitz transformation [5, III-14]

$$\begin{aligned} &{}_3\Phi_2\left[\begin{matrix} q^{-k}, & \mu, & q^k\lambda \\ & q^{1-k}/\mu, & q^{1-2k}/\lambda \end{matrix} \middle| q; -q^{1-2k}x/\lambda\mu y\right] \\ &= \frac{(-q^{-k}x/y;q)_\infty}{(-x/y;q)_\infty} {}_5\Phi_4\left[\begin{matrix} q^{-\frac{k}{2}}, -q^{-\frac{k}{2}}, q^{\frac{1-k}{2}}, -q^{\frac{1-k}{2}}, q^{1-2k}/\lambda\mu \\ q^{1-k}/\mu, q^{1-2k}/\lambda, -q^{-k}x/y, -qy/x \end{matrix} \middle| q; q\right], \end{aligned} \tag{3.5}$$

we can derive the following general transformation formula.

Theorem 11 *(q-Analogue of transformation [3, Eq. (2.1)]). For an arbitrary complex sequence $\{\Omega(n)\}_{n=0}^{\infty}$, there holds the following transformation*

$$\sum_{m,n=0}^{\infty} q^{mn+\binom{n+1}{2}} \Omega(m+n)(\lambda;q)_{2m+n}(\lambda;q)_{m+2n}(\mu;q)_m(\mu;q)_n \frac{x^m}{(q;q)_m} \frac{y^n}{(q;q)_n} \tag{3.6a}$$

$$= \sum_{m,n=0}^{\infty} q^{n^2} \Omega(m+2n)(\lambda\mu;q)_{2m+4n} \frac{(\lambda;q)_{m+2n}(\lambda;q)_{2m+3n}(\mu;q)_{m+n}}{(\lambda\mu;q)_{2m+3n}} \tag{3.6b}$$

$$\times \frac{(-q^{1+n}y/x;q)_m}{(q;q)_m} \frac{(-y)^n}{(q;q)_n} x^{m+n} \tag{3.6c}$$

provided that both double series displayed above are absolutely convergent.

Proof. Setting $n = k - m$, we can rewrite (3.6a) as follows:

$$\sum_{k=0}^{\infty} \Omega(k)(\lambda;q)_k(\lambda;q)_{2k}(\mu;q)_k \frac{q^{\binom{k+1}{2}} y^k}{(q;q)_k} {}_3\Phi_2\left[\begin{matrix} q^{-k}, \mu, q^k\lambda \\ q^{1-k}/\mu, q^{1-2k}/\lambda \end{matrix} \middle| q; \frac{-q^{1-2k}x}{\lambda\mu y}\right].$$

Evaluating the ${}_3\Phi_2$-series by the Sears–Carlitz transformation (3.5) and then simplifying the result, we find that the expression (3.6a) becomes

$$\sum_{k=0}^{\infty} \Omega(k)(\lambda;q)_k \sum_{n=0}^{[\frac{k}{2}]} (-y)^n x^{k-n} q^{n^2} \frac{(\lambda;q)_{2k-n}(\mu;q)_{k-n}(q^{2k-n}\lambda\mu;q)_n(-qy/x;q)_{k-n}}{(q;q)_n(q;q)_{k-2n}(-qy/x;q)_n},$$

where $[s]$ denotes the greatest integer $\leqslant s$.

Finally, going back to a series

$$\sum_{m,n=0}^{\infty} q^{n^2}\Omega(m+2n)(\lambda\mu;q)_{2m+4n}\frac{(\lambda;q)_{m+2n}(\lambda;q)_{2m+3n}(\mu;q)_{m+n}}{(\lambda\mu;q)_{2m+3n}} \times \frac{(-q^{1+n}y/x;q)_m}{(q;q)_m}\frac{(-y)^n}{(q;q)_n}x^{m+n}$$

by setting $k=m+2n$, we just get the series transformation stated in the theorem. □

Our next series identity reads as follows.

Theorem 12 *(q-Analogue of reduction formula [3, Eq. (2.10)]). For an arbitrary complex sequence $\{\Omega(n)\}_{n=0}^{\infty}$, there holds the following transformation*

$$\sum_{m,n=0}^{\infty}(-1)^m q^{\binom{n}{2}}\Omega(m+n)\frac{(\lambda;q)_{2m+n}(q\lambda/uv;q)_m x^{m+n}}{(q\lambda/u;q)_m(q\lambda/v;q)_m(q;q)_m(q;q)_n} \tag{3.7a}$$

$$=\sum_{n=0}^{\infty}\Omega(n)q^{\binom{n+1}{2}}\frac{(\lambda;q)_n(u;q)_n(v;q)_n}{(q;q)_n(q\lambda/u;q)_n(q\lambda/v;q)_n}\left(\frac{\lambda x}{uv}\right)^n \tag{3.7b}$$

provided that both series displayed above are absolutely convergent.

Proof. Recalling the q-Pfaff–Saalschütz theorem [5, II-12]

$${}_3\Phi_2\left[\begin{matrix} q^{-n}, & a, & b \\ & c, & q^{1-n}ab/c \end{matrix}\,\middle|\, q;q\right]=\frac{[c/a,c/b;q]_n}{[c,c/ab;q]_n}$$

the expression (3.7a) may be written

$$\sum_{n=0}^{\infty}\Omega(n)q^{\binom{n}{2}}\frac{(\lambda;q)_n}{(q;q)_n}x^n\,{}_3\Phi_2\left[\begin{matrix} q^{-n}, & q^n\lambda, & q\lambda/uv \\ & q\lambda/u, & q\lambda/v \end{matrix}\,\middle|\, q;q\right]$$

$$=\sum_{n=0}^{\infty}\Omega(n)q^{\binom{n+1}{2}}\frac{(\lambda;q)_n(u;q)_n(v;q)_n}{(q\lambda/u;q)_n(q\lambda/v;q)_n(q;q)_n}\left(\frac{\lambda x}{uv}\right)^n,$$

which coincides with (3.7b). This completes the proof of the theorem. □

Setting $\Omega(n)=1$ in Theorem 12, we recover the Sears–Carlitz transformation (3.5). Similarly by means of Jackson's q-Dougall–Dixon formula [5, II-22]

$${}_8\Phi_7\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^{-n} \\ & \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & aq^{n+1} \end{matrix}\,\middle|\, q;q\right]$$

$$=\frac{[qa,qa/bc,qa/bd,qa/cd;q]_n}{[qa/b,qa/c,qa/d,qa/bcd;q]_n}\quad \text{where } q^{n+1}a^2=bcde,$$

we can deduce the following transformation.

Theorem 13 *(q-Analogue of reduction formula [3, Eq. (3.8)]). For an arbitrary complex sequence $\{\Omega(n)\}_{n=0}^{\infty}$, there holds the following transformation*

$$\sum_{m,n=0}^{\infty} \Omega(m+n) \frac{(q\lambda^2/uvw;q)_{2m+n} x^{m+n}}{(q\lambda;q)_{2m+n}(q\lambda^2/uvw;q)_{m+n}(q\lambda/uvw;q)_{m+n}}$$
$$\times \frac{[\lambda, q\lambda^{\frac{1}{2}}, -q\lambda^{\frac{1}{2}}, u, v, w; q]_m}{[q, \lambda^{\frac{1}{2}}, -\lambda^{\frac{1}{2}}, q\lambda/u, q\lambda/v, q\lambda/w; q]_m} \frac{(q\lambda/uvw;q)_n}{(q;q)_n} \left(\frac{q\lambda}{uvw}\right)^m$$
$$= \sum_{n=0}^{\infty} \Omega(n) \frac{[q\lambda/uv, q\lambda/uw, q\lambda/vw; q]_n}{[q\lambda/u, q\lambda/v, q\lambda/w, q\lambda/uvw; q]_n} \frac{x^n}{(q;q)_n}$$

provided that both series displayed above are absolutely convergent.

References

[1] R.G. Buschman, H.M. Srivastava, Series identities and reducibility of Kampé de Fériet functions, Math. Proc. Cambridge Philos. Soc. 91 (1982) 435–440.

[2] W.-C.C. Chan, K.-Y. Chen, C.-J. Chyan, H.M. Srivastava, Some multiple hypergeometric transformations and associated reduction formulas, J. Math. Anal. Appl. 294 (2004) 418–437.

[3] K.-Y. Chen, H.M. Srivastava, Series identities and associated families of generating functions, J. Math. Anal. Appl. 311 (2005) 582–599.

[4] W.-C. Chu, H.M. Srivastava, Ordinary and basic bivariate hypergeometric transformations associated with the Appell and Kampé de Fériet functions, J. Comput. Appl. Math. 156 (2003) 355–370.

[5] G. Gasper, M. Rahman, Basic Hypergeometric Series, Cambridge Univ. Press, Cambridge, 1990.

[6] Per W. Karlsson, Transformations of multiple q-series with quasi-arbitrary terms, J. Math. Anal. Appl. 231 (1999) 241–254.

[7] H.M. Srivastava, V.K. Jain, q-Series identities and reducibility of basic double hypergeometric functions, Canad. J. Math. 38 (1986) 215–231.

[8] H.M. Srivastava, P.W. Karlsson, Multiple Gaussian Hypergeometric Series, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, London and Toronto, 1985.